Stability of the Prékopa-Leindler inequality

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Abstract

We prove a stability version of the Prékopa-Leindler inequality.

1 The problem

Our main theme is the Prékopa-Leindler inequality, due to A. Prékopa [14] and L. Leindler [13]. Soon after its proof, the inequality was generalized in A. Prékopa [15] and [16], C. Borell [7], and in H.J. Brascamp, E.H. Lieb [8]. Various applications are provided and surveyed in K.M. Ball [1], F. Barthe [5], and R.J. Gardner [12]. The following multiplicative version from [1], is often more useful and is more convenient for our purposes.

THEOREM 1.1 (Prékopa-Leindler) *If m, f, g are non-negative integrable functions on $\mathbb{R}$ satisfying $m(\frac{r+s}{2}) \geq \sqrt{f(r)g(s)}$ for $r, s \in \mathbb{R}$, then*

$$\int_{\mathbb{R}} m \geq \sqrt{\int_{\mathbb{R}} f \cdot \int_{\mathbb{R}} g}.$$

S. Dubuc [9] characterized the equality case if the integrals of f, g, m above are positive. For this characterization, we say that a non-negative real function h on $\mathbb{R}$ is log-concave if for any $x, y \in \mathbb{R}$ and $\lambda \in (0,1)$, we have

$$h((1-\lambda)x + \lambda y) \geq h(x)^{1-\lambda} h(y)^{\lambda}.$$

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In other words, the support of h is an interval, and $\log h$ is concave on the support. Now [9] proved that equality holds in the Prékopa-Leindler inequality if and only if there exist $a > 0$, $b \in \mathbb{R}$ and a log-concave h with positive integral on $\mathbb{R}$ such that for a.e. $t \in \mathbb{R}$, we have

$$\begin{aligned} m(t) &= h(t) \\ f(t) &= a \cdot h(t+b) \\ g(t) &= a^{-1} \cdot h(t-b). \end{aligned}$$

In addition for all $t \in R$, we have $m(t) \geq h(t)$, $f(t) \leq a \cdot h(t+b)$ and $g(t) \leq a^{-1} \cdot h(t-b)$.

Our goal is to prove a stability version of the Prékopa-Leindler inequality.

THEOREM 1.2 *There exists an positive absolute constant c with the following property. If m, f, g are non-negative integrable functions with positive integrals on $\mathbb{R}$ such that m is log-concave, $m(\frac{r+s}{2}) \geq \sqrt{f(r)g(s)}$ for $r, s \in \mathbb{R}$, and*

$$\int_{\mathbb{R}} m \leq (1+\varepsilon)\sqrt{\int_{\mathbb{R}} f \cdot \int_{\mathbb{R}} g},$$

for $\varepsilon > 0$, then there exist $a > 0$, $b \in \mathbb{R}$ such that

$$\begin{aligned} \int_{\mathbb{R}} |f(t) - a\, m(t+b)|\, dt &\leq c \cdot \sqrt[3]{\varepsilon}|\ln \varepsilon|^{\frac{4}{3}} \cdot a \cdot \int_{\mathbb{R}} m(t)\, dt \\ \int_{\mathbb{R}} |g(t) - a^{-1} m(t-b)|\, dt &\leq c \cdot \sqrt[3]{\varepsilon}|\ln \varepsilon|^{\frac{4}{3}} \cdot a^{-1} \cdot \int_{\mathbb{R}} m(t)\, dt. \end{aligned}$$

REMARK 1.3 *The statement also holds if the condition that m is log concave, is replaced by the condition that both f and g are log-concave. The reason is that the function*

$$\tilde{m}(t) = \sup\{\sqrt{f(r)g(s)} : \, t = \tfrac{r+s}{2}\}$$

is log-concave in this case.

REMARK 1.4 *Most probably, the optimal error estimate in Theorem 1.2 is of order ε. This cannot be proved using the method of this note; namely, by proving first an estimate on the quadratic transportation distance.*

Let us summarize the main idea to prove Theorem 1.2. It can be assumed that f and g are log-concave probability distributions with zero mean (see Section 6). We establish the main properites of log-concave functions in Section 2, and introduce the transportation map in Section 3. After translating the condition $\int_{\mathbb{R}} m \leq (1+\varepsilon)\sqrt{\int_{\mathbb{R}} f \cdot \int_{\mathbb{R}} g}$ into an estimate for the transportation map, we estimate the quadratic transportation distance in Section 4. Based on this, we estimate the L_1 distance of f and g in Section 5, which leads to the proof Theorem 1.2 in Section 6. We note that the upper bound in Section 5 for the L_1 distance of two log-concave probability distributions in terms of the their quadratic transportation distance is close to being optimal.

REMARK 1.5 *It is not clear whether the condition in Theorem 1.2 that m is log-concave is necessary for there to be a stability estimate.*

REMARK 1.6 *Given $\alpha, \beta \in (0,1)$ with $\alpha + \beta = 1$, we also have the following version of the Prékopa-Leindler inequality: If m, f, g are non-negative integrable functions on $\mathbb{R}$ satisfying $m(\alpha r + \beta s) \geq f(r)^\alpha g(s)^\beta$ for $r, s \in \mathbb{R}$, then*

$$\int_{\mathbb{R}} m \geq \left(\int_{\mathbb{R}} f\right)^\alpha \left(\int_{\mathbb{R}} g\right)^\beta.$$

The method of this note also yields the corresponding stability estimate, only the c in the new version of Theorem 1.2 depends on α. For this statement, the formula

$$\frac{1+T'(x)}{2\sqrt{T'(x)}} = 1 + \frac{(1-T'(x))^2}{2\sqrt{T'(x)}(1+\sqrt{T'(x)})^2},$$

used widely in this note if $T'(x)$ is "not too large", should be replaced with Koebe's estimate

$$\frac{\alpha + \beta T'(x)}{T'(x)^\beta} \geq 1 + \frac{\min\{\alpha, \beta\}(1-T'(x))^2}{T'(x)^\beta(1+\sqrt{T'(x)})^2}.$$

In addition, if $T'(x)$ is "large", then one uses $\frac{\alpha+\beta T'(x)}{T'(x)^\beta} > \beta T'(x)^\alpha$.

REMARK 1.7 *The Prékopa-Leindler inequality also holds in higher dimensions. One possible approach to have a higher dimensional analogue of the stability statement is to use Theorem 1.2 and a suitable stability version of the injectivity of the Radon transform on log-concave functions. Here the*

difficulty is caused by the fact that the Radon transform is notoriously instable even on the space of smooth functions. Another possible approach is to use recent stability version of the Brunn-Minkowski inequality due to A. Figalli, F. Maggi, A. Pratelli [10] and [11]. This approach has been successfully applied in K.M. Ball, K.J. Böröczky [4].

2 Some elementary properties of log-concave probability distributions on $\mathbb{R}$

Let h be a log-concave probability distribution on $\mathbb{R}$. In this section we discuss various useful elementary properties of h. Many of these properties are implicit or explicit in [2].

First, assuming $h(t_0) = a \cdot b^{t_0}$ for $a, b > 0$, and $t_1 < t_0 < t_2$, we have

$$\begin{array}{l} \text{if } h(t_1) \geq a \cdot b^{t_1}, \text{ then } h(t_2) \leq a \cdot b^{t_2}, \\ \text{if } h(t_2) \geq a \cdot b^{t_2}, \text{ then } h(t_1) \leq a \cdot b^{t_1}. \end{array} \tag{1}$$

Next we write w_h and μ_h to denote the median and mean of h; namely,

$$\int_{-\infty}^{w_h} h = \int_{w_h}^{\infty} h = \frac{1}{2} \quad \text{and} \quad \mu_h = \int_{\mathbb{R}} x h(x)\, dx.$$

PROPOSITION 2.1 *If f and g are positive, and θ is an increasing function on (a, b), and there exists $c \in (a, b)$ such that $f(t) \leq g(t)$ if $t \in (a, c)$, and $f(t) \geq g(t)$ if $t \in (c, b)$, and $\int_a^b g(t)\, dt = \int_a^b f(t)\, dt$ then*

$$\int_a^b \theta(t) g(t)\, dt \leq \int_a^b \theta(t) f(t)\, dt.$$

Proof: We may assume that $g(t) > 0$ if $t \in (a, c)$, and $f(t) > 0$ if $t \in (c, b)$. Let (a_0, b) and (a, b_0) be the support of f and g, respectively, where $a_0 \in [a, c]$ and $b_0 \in [c, b]$. Let $T : (a_0, b) \to (a, b_0)$ be the transportation map defined by

$$\int_{a_0}^{x} f(t)\, dt = \int_a^{T(x)} g(t)\, dt.$$

The conditions yield that T is monotone increasing, bijective, continuous, $T(x) \leq x$ for $x \in (a_0, b)$, and for a.e. $x \in (a_0, b)$, we have

$$f(x) = g(T(x)) T'(x).$$

Therefore

$$\begin{aligned}\int_a^b \theta(t)g(t)\,dt &= \int_{a_0}^b \theta(T(s))g(T(s))T'(s)\,ds = \int_{a_0}^b \theta(T(s))f(s)\,ds \\ &\leq \int_a^b \theta(s)f(s)\,ds. \ \square\end{aligned}$$

PROPOSITION 2.2 *If h is a log-concave probability distribution on $\mathbb{R}$ then for $w = w_h$ and $\mu = \mu_h$, we have*

(i) $h(w) \cdot |w - \mu| \leq \ln \sqrt{e/2}$.

(ii) $h(w) \cdot e^{-2h(w)|x-w|} \leq h(x) \leq h(w) \cdot e^{2h(w)|x-w|}$ *if* $|x - w| \leq \frac{\ln 2}{2h(w)}$.

(iii) $h(x) \leq 2h(w)$ *for* $x \in \mathbb{R}$.

(iv) If $x > w$ *then* $\int_x^\infty h \leq \frac{h(x)}{2h(w)}$.

(v) If $x > w$ *and* $\int_x^\infty h = \nu > 0$, *then*

$$\begin{aligned}\int_x^\infty (t - w)h(t)\,dt &\leq \frac{\nu}{4h(w)} \cdot (1 - \ln 2\nu) \\ \int_x^\infty (t - w)^2 h(t)\,dt &\leq \frac{\nu}{8h(w)^2} \cdot [(\ln 2\nu)^2 - 2\ln 2\nu + 2].\end{aligned}$$

Remark All estimates are optimal.

Proof: We may assume that $w_h = 0$, and $h(w) = \frac{1}{2}$. It is natural to compare h near 0 to the probability distribution

$$\varphi(x) = \begin{cases} \frac{1}{2} \cdot e^{-x} & \text{if } x \geq -\ln 2 \\ 0 & \text{if } x < -\ln 2, \end{cases}$$

which satisfies $w_\varphi = 0$, and $\varphi(0) = h(0)$. Since $\int_{-\infty}^0 h = \int_{-\infty}^0 \varphi$, we have $h(x) \geq \varphi(x)$ for some $x > 0$. It follows from (1) that there exists some $v > 0$ such that

$$\begin{aligned} h(x) &\geq \varphi(x) & &\text{provided } x \in [0, v] \\ h(x) &\leq \varphi(x) & &\text{provided } x \geq v \text{ or } x \in [-\ln 2, 0]. \end{aligned} \tag{2}$$

In particular $\int_{-\infty}^{0} h = \int_{-\infty}^{0} \varphi$ and $\int_{0}^{\infty} h = \int_{0}^{\infty} \varphi$ yield

$$\begin{aligned} -\ln\frac{e}{2} &= \int_{-\infty}^{0} x\varphi(x)\,dx + \int_{0}^{\infty} x\varphi(x)\,dx \\ &\le \int_{-\infty}^{0} xh(x)\,dx + \int_{0}^{\infty} xh(x)\,dx = \mu. \end{aligned}$$

Comparing h to $\varphi(-x)$ shows that $\mu \le \ln\frac{e}{2}$, and in turn, we deduce (i).

Turning to (ii), the upper bound directly follows from (2). To prove the lower bound, we may assume that $x > 0$. According to (2), it is enough to check the case $x = \ln 2$. Therefore we suppose that

$$h(\ln 2) < 1/4,$$

and seek a contradiction. Since h is log-concave, there exists some $a \in \mathbb{R}$ such that

$$h(x) < \tfrac{1}{4}\, e^{-a(x-\ln 2)} \quad \text{for } x \in \mathbb{R}.$$

Here $h(0) = \frac{1}{2}$ yields that $a \ge 1$.

We observe that $\frac{1}{4}\, e^{a(x_0-\ln 2)} = \frac{1}{2}\, e^{x_0}$ for $x_0 = \frac{a-1}{a+1}\ln 2$, and applying the analogue of (2) to $\varphi(-x)$, we obtain that $h(x) \le \frac{1}{2}\, e^{x}$ for $x \in [0, x_0]$. In particular

$$\int_0^\infty h < \int_0^{x_0} \frac{1}{2}\, e^{x} dx + \int_{x_0}^{\infty} \frac{1}{4}\, e^{-a(x-\ln 2)} dx = \left(\frac{1}{a} + 1\right) 2^{-\frac{2}{a+1}} - \frac{1}{2}.$$

Differentiation shows that the last expression is first decreasing, and after increasing in $a \ge 1$. Since the value of this last expression is $\frac{1}{2}$ both at $a = 1$ and at $a = \infty$, we deduce that $\int_0^\infty h < \frac{1}{2}$. This is absurd, therefore we have proved (ii).

To prove (iii), we may assume $x > 0$ and $h(x) \ge 1$. Since $h(t) \ge \frac{1}{2}\, e^{\frac{t}{x}\,\ln 2h(x)}$ for $t \in [0, x]$, we have

$$\frac{1}{2} \ge \int_0^x h \ge \int_0^x \frac{1}{2}\, e^{\frac{t}{x}\,\ln 2h(x)} dt = \frac{x(2h(x)-1)}{2\ln 2h(x)}.$$

Now (ii) and $h(x) \ge 1$ yield that $x \ge \ln 2$. As $\frac{s-1}{s} > \frac{1}{\ln 2}$ for $s > 2$, we conclude $h(x) \le 1$.

To prove (iv), we may assume that $h(x) < h(w)$. Let $x_0 = -\ln 2h(x)$, and hence $h(x) = \frac{1}{2}\,e^{-x_0}$. If $h(x) \geq h(v)$, then (2) yields $\int_0^x h(t)\,dt \geq \int_0^{x_0} \frac{1}{2}\,e^{-t}\,dt$, and hence

$$\int_x^\infty h(t)\,dt \leq \int_{x_0}^\infty \frac{1}{2}\,e^{-t}\,dt = h(x).$$

If $h(x) < h(v)$ then $x_0 \geq x$. We may assume that $x_0 > x$, and hence $h(x) < \frac{1}{2}\,e^{-x}$. We choose $a > 0$ such that

$$\int_x^\infty h(t)\,dt = \int_x^\infty h(x)\,e^{-a(t-x)}\,dt,$$

and consider the function

$$\tilde{h}(t) = \begin{cases} h(t) & \text{if } t \leq x \\ h(x)\,e^{-a(t-x)} & \text{if } t \geq x, \end{cases}$$

If follows by the choice of a that $h(t) \geq \tilde{h}(t)$ for some $t > x$. We deduce that $\tilde{h}$ is also log-concave, and hence $\tilde{h}(t) \leq \frac{1}{2}\,e^{-t}$ for $t \geq v$. Therefore $a \geq 1$, and we conclude that

$$\int_x^\infty h(t)\,dt = \int_x^\infty h(x)\,e^{-a(t-x)}\,dt = h(x)/a \leq h(x).$$

Finally, we prove (v). Let $x_1 = -\ln 2\nu$, which satisfies that $\int_x^\infty h(t)\,dt = \int_{x_1}^\infty \frac{1}{2}\,e^{-t}\,dt$. It follows from (2) that $x_1 \geq x$. We define two functions f and g on $[x, \infty)$. Let $f(t) = \frac{1}{2}\,e^{-t}$ if $t \geq x_1$, and let $f(t) = 0$ if $t \in [x, x_1)$. In addition let $g = h|_{[x,\infty)}$. These two functions satisfy the conditions in Proposition 2.1, therefore for $\alpha \geq 0$, we have

$$\int_x^\infty t^\alpha h(t)\,dt = \int_x^\infty t^\alpha g(t)\,dt \leq \int_x^\infty t^\alpha f(t)\,dt = \int_{x_1}^\infty \frac{t^\alpha e^{-t}}{2}\,dt.$$

Evaluating the last integral for $\alpha = 1, 2$ yields (v). $\square$

Next we discuss various consequences of Proposition 2.2.

COROLLARY 2.3 *Let h be a log-concave probability density function on $\mathbb{R}$, and let $\int_x^\infty h = \nu \in (0, \frac{1}{2}]$. Then*

(i) $h(x) \cdot e^{-\frac{h(x)|t-x|}{\nu}} \leq h(t) \leq h(x) \cdot e^{\frac{h(x)|t-x|}{\nu}}$ *if* $|t - x| \leq \frac{\nu \ln 2}{h(x)}$*;*

(ii) If $\nu \in (0, \frac{1}{6})$, $w = w_h$ and $\mu = \mu_h$, then

$$
\begin{aligned}
\int_x^\infty |t-\mu| h(t)\, dt &\leq \frac{\nu}{2h(w)} \cdot |\ln \nu| \\
\int_x^\infty |t-\mu|^2 h(t)\, dt &\leq \frac{5\nu}{4h(w)^2} \cdot (\ln \nu)^2.
\end{aligned}
$$

Remark The order of all estimates is optimal, as it is shown by the example of $h(t) = e^{-|t|}/2$.

Proof: To prove (i), let $|t - x| \leq \frac{\nu \ln 2}{h(x)}$. There exists a unique $\lambda \in \mathbb{R}$, such that for the function

$$
\tilde{h}(t) = \begin{cases} h(t) & \text{if } t \geq x; \\ \min\{h(t), h(x) \cdot e^{\lambda(t-x)}\} & \text{if } t \leq x. \end{cases},
$$

we have $\int_{-\infty}^x \tilde{h} = \nu$. We note that $\tilde{h}$ is log-concave, and $\lambda \geq \frac{-h(x)}{\nu}$. In particular $\frac{1}{2\nu}\tilde{h}$ is a log-concave probability distribution whose median is x, and hence Proposition 2.2 (ii) yields $h(t) \geq \tilde{h}(t) \geq h(x) \cdot e^{\frac{-h(x)|t-x|}{\nu}}$. Since for $s = 2x - t$, we have $h(s) \geq h(x) \cdot e^{\frac{-h(x)|s-x|}{\nu}}$, we conclude (i) by (1).

For (ii), we may assume that $h(w) = \frac{1}{2}$, and hence Proposition 2.2 (i) yields that $|w - \mu| \leq \ln \frac{e}{2}$. Since $\ln 2\nu \leq -1$, we deduce by Proposition 2.2 (ii) that

$$
\begin{aligned}
\int_x^\infty |t-\mu| h(t)\, dt &\leq \int_x^\infty [|t-w| + |w-\mu|] h(t)\, dt \\
&\leq \nu \cdot (-\ln 2\nu) + \nu \cdot \ln \tfrac{e}{2} < \nu \cdot |\ln \nu|.
\end{aligned}
$$

In addition

$$
\begin{aligned}
\int_x^\infty (t-\mu)^2 h(t)\, dt &\leq \int_x^\infty 2[(t-w)^2 + (w-\mu)^2] h(t)\, dt \\
&\leq \nu \cdot 5(\ln 2\nu)^2 + \nu \cdot 2(\ln \tfrac{e}{2})^2 < 5\nu \cdot (\ln \nu)^2. \;\square
\end{aligned}
$$

3 The transportation map for log-concave probability distributions, and the Prékopa-Leindler inequality

Let f and g be log-concave probability distributions on $\mathbb{R}$, and let I_f and I_g denote the open intervals that are the supports of f and g, respectively. We define the transportation map $T: I_f \to I_g$ by the identity

$$\int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{T(x)} g(t)\,dt. \tag{3}$$

In particular T is monotone increasing, bijective, and continuous on I_f, and for a.e. $x \in I_f$, we have

$$f(x) = g(T(x))T'(x). \tag{4}$$

Remark Using (3), the transportation map $T : \mathbb{R} \to \mathbb{R}$ can be defined for any two probability distributions f and g, and T is naturally monotone increasing. It is easy to see that (4) holds if there exists a set $A \subset \mathbb{R}$ of zero measure such that both f and g are continous on $\mathbb{R}\backslash A$. Unfortunately (4) does not hold in general. Let $B \subset \mathbb{R}$ be such a set that the density of each point of B is strictly between 0 and 1, and let f be a probability distribution that is zero on $\mathbb{R}\backslash B$, and positive and continuous on B. If say $g(x) = \frac{1}{2}\,e^{|x|}$, then (4) never holds.

For an integrable function m on $\mathbb{R}$ satisfying $m(\frac{r+s}{2}) \geq \sqrt{f(r)g(s)}$ for $r, s \in \mathbb{R}$, one proof of the Prékopa-Leindler inequality runs as follows:

$$\begin{aligned}
1 &= \int_{\mathbb{R}} f = \int_{I_f} \sqrt{f(x)} \cdot \sqrt{g(T(x))T'(x)}\,dx \\
&\leq \int_{I_f} m\left(\frac{x+T(x)}{2}\right) \sqrt{T'(x)}\,dx \\
&\leq \int_{I_f} m\left(\frac{x+T(x)}{2}\right) \cdot \frac{1+T'(x)}{2}\,dx \\
&= \int_{\frac{1}{2}(I_f+I_g)} m(x)\,dx \leq \int_{\mathbb{R}} m.
\end{aligned}$$

The basic fact that we will exploit is this. If we know that $\int_{\mathbb{R}} m \leq 1+\varepsilon$ then

$$\begin{aligned}
\varepsilon &\geq \int_{I_f} m\left(\frac{x+T(x)}{2}\right)\cdot\left(\frac{1+T'(x)}{2}-\sqrt{T'(x)}\right)dx \\
&\geq \int_{I_f} \sqrt{f(x)}\cdot\sqrt{g(T(x))T'(x)}\left(\frac{1+T'(x)}{2\sqrt{T'(x)}}-1\right)dx \\
&= \int_{I_f} f(x)\cdot\frac{(1-\sqrt{T'(x)})^2}{2\sqrt{T'(x)}}\,dx. \qquad (5)
\end{aligned}$$

As long as T' is not too large, the integrand is at least about $f(x)(1-T'(x))^2$ and using a Poincaré inequality for the density f we can bound the integral of this expression from below by the transportation cost $\int f(x)(x-T(x))^2$. The main technical issue is to handle the places where T' is large.

4 The quadratic transportation distance

Let f and g be log-concave probability distributions on $\mathbb{R}$ with zero mean; namely,

$$0=\int_{\mathbb{R}} xf(x)\,dx=\int_{\mathbb{R}} yg(y)\,dy.$$

In this section we show that (5) yields an upper bound for the quadratic transportation distance

$$\int_{I_f} f(x)(T(x)-x)^2dx$$

of f and g.

LEMMA 4.1 *If f and g are log-concave probability distributions on $\mathbb{R}$ with zero mean, and (5) holds for $\varepsilon\in(0,\frac{1}{48})$, then*

$$\int_{I_f} f(x)(T(x)-x)^2dx \leq 2^{20}f(w_f)^{-2}\cdot\varepsilon|\ln\varepsilon|^2.$$

Remark The optimal power of ε is most probably ε^2 in Lemma 4.1 (compare Example 7.1). For a possible proof of an improved estimate, we should

improve on (6) if $R(x) = T(x) - x$ where T is the transportation map for another log-concave probability distribution. One may possibly use that $T(x) - x$ is of at most logarithmic order.

Proof: The main tool in the proof of Lemma 4.1 is the Poincaré inequality for log-concave measures which can be found in (1.3) and (4.2) of S.G. Bobkov [6]. If h is a log-concave probability distribution on $\mathbb{R}$, and the function R on $\mathbb{R}$ is locally Lipschitz with expectation $\mu = \int_{\mathbb{R}} h(x)R(x)\,dx$, then

$$\int_{\mathbb{R}} h(x)(R(x)-\mu)^2\,dx = \int_{\mathbb{R}} h(x)R(x)^2\,dx - \mu^2 \leq h(w_h)^{-2} \cdot \int_{\mathbb{R}} h(x)R'(x)^2\,dx. \tag{6}$$

We may assume that $g(w_g) \leq f(w_f)$, and $f(w_f) = \frac{1}{2}$. Let T be the transportation map from f to g, and let S be its inverse, thus for a.e. $x \in I_f$ and $y \in I_g$, we have

$$f(x) = g(T(x))T'(x) \quad \text{and} \quad g(y) = f(S(y))S'(y). \tag{7}$$

Suppose that for some $x \in \mathbb{R}$ with $\int_x^\infty f = \nu \in (0, \frac{1}{2}]$, we have $g(T(x)) \leq \frac{1}{16}\,f(x)$. If $x \leq t \leq x + \frac{\nu \ln 2}{f(x)}$ then Corollary 2.3 (i) yields $f(t) \geq f(x) \cdot e^{-\frac{f(x)(t-x)}{\nu}} \geq \frac{1}{2}\,f(x)$. On the other hand, the log-concavity of g and Proposition 2.2 (iii) yield that if $x \leq t < x + \frac{\nu \ln 2}{f(x)}$, then $g(t) < 2g(x) \leq \frac{1}{4}\,f(t)$. In particular $T'(t) > 4$ by (7), and hence (compare (5))

$$\varepsilon \geq \int_{\mathbb{R}} \frac{(1-\sqrt{T'(t)})^2}{2\sqrt{T'(t)}}\,f(t)\,dt > \int_x^{x+\frac{\nu \ln 2}{f(x)}} \frac{f(x)}{4} \cdot e^{-\frac{f(x)(t-x)}{\nu}}\,dt = \frac{\nu}{8}.$$

Similar argument for $f(-x)$ and $g(-x)$ shows that if $\int_{-\infty}^x f = \nu$ and $g(T(x)) \leq \frac{1}{16}\,f(x)$ then $\nu < 8\varepsilon$.

We define x_1, x_2, y_1, y_2 by

$$\int_{-\infty}^{x_1} f = \int_{x_2}^{\infty} f = \int_{-\infty}^{y_1} g = \int_{y_2}^{\infty} g = 8\varepsilon < \frac{1}{6}.$$

The argument above yields that if $x \in (x_1, x_2)$, then $T'(x) \leq 16$ and $g(T(x)) \geq \frac{1}{16}\,f(x)$, and hence $g(w_g) \geq \frac{1}{32}$. As the means of f and g are zero, we deduce

by Corollary 2.3 (ii) and (7) that

$$\int_{\mathbb{R}\backslash[x_1,x_2]} |x|f(x)\,dx \leq 2^4\varepsilon|\ln\varepsilon|; \tag{8}$$

$$\int_{\mathbb{R}\backslash[x_1,x_2]} |T(x)|f(x)\,dx = \int_{\mathbb{R}\backslash[y_1,y_2]} |y|g(y)\,dy \leq 2^8\varepsilon|\ln\varepsilon|; \tag{9}$$

$$\int_{\mathbb{R}\backslash[x_1,x_2]} x^2 f(x)\,dx \leq 2^7\varepsilon(\ln\varepsilon)^2; \tag{10}$$

$$\int_{\mathbb{R}\backslash[x_1,x_2]} T(x)^2 f(x)\,dx = \int_{\mathbb{R}\backslash[y_1,y_2]} y^2 g(y)\,dy \leq 2^{15}\varepsilon(\ln\varepsilon)^2. \tag{11}$$

Since $(T(x)-x)^2 \leq 2[T(x)^2 + x^2]$, we have

$$\int_{\mathbb{R}\backslash[x_1,x_2]} (T(x)-x)^2 f(x)\,dx \leq 2^{17}\varepsilon(\ln\varepsilon)^2. \tag{12}$$

Next we consider the log-concave probability distribution

$$\tilde{f}(t) = \begin{cases} (1-16\varepsilon)^{-1} f(t) & \text{if } t \in [x_1,x_2] \\ 0 & \text{if } t \in \mathbb{R}\backslash[x_1,x_2]. \end{cases}$$

To estimate $\tilde{f}(w_{\tilde{f}})$, we define $z_1 = w_f - \ln 2$, and $z_2 = w_f + \ln 2$. Since $f(w_f) = \frac{1}{2}$, Proposition 2.2 (ii) applied to f yields

$$\int_{\mathbb{R}\backslash[z_1,z_2]} \tilde{f}(x)\,dx \leq (1-16\varepsilon)^{-1}\left(1 - 16\varepsilon - \int_{z_1}^{z_2} \frac{e^{-|x-w_f|}}{2}\,dx\right) < \frac{1}{2}.$$

It follows that $|w_{\tilde{f}} - w_f| < \ln 2$, and hence we deduce again by Proposition 2.2 (ii) that

$$\tilde{f}(w_{\tilde{f}}) > \frac{1}{4}.$$

For the expectation

$$\mu = \int_{\mathbb{R}} (T(x)-x)\tilde{f}(x)\,dx,$$

we have the estimate

$$|\mu| = (1-16\varepsilon)^{-1}\left|\int_{\mathbb{R}\backslash[x_1,x_2]} (T(x)-x)f(x)\,dx\right| \leq 2^{10}\varepsilon|\ln\varepsilon|.$$

If $x \in (x_1, x_2)$, then $T'(x) \leq 16$, thus the expression in (5) satisfies

$$\frac{(1-\sqrt{T'(x)})^2}{2\sqrt{T'(x)}} = \frac{(T'(x)-1)^2}{2(1+\sqrt{T'(x)})^2\sqrt{T'(x)}} \geq \frac{(T'(x)-1)^2}{200} > 2^{-8}(T'(x)-1)^2.$$

We deduce using (6) and (5) that

$$\begin{aligned}
\int_{[x_1,x_2]} (T(x)-x)^2 f(x)\,dx &\leq \int_{\mathbb{R}} (T(x)-x)^2 \tilde{f}(x)\,dx \\
&\leq \mu^2 + \tilde{f}(w_{\tilde{f}})^{-2} \int_{\mathbb{R}} (T'(x)-1)^2 \tilde{f}(x)\,dx \\
&\leq 2^{20}\varepsilon^2 |\ln\varepsilon|^2 + 2^{13} \int_{x_1}^{x_2} \frac{(1-\sqrt{T'(x)})^2}{2\sqrt{T'(x)}}\, f(x)\,dx \\
&\leq 2^{20}\varepsilon^2 |\ln\varepsilon|^2 + 2^{13}\varepsilon. \qquad (13)
\end{aligned}$$

However (10) and (11) imply

$$\int_{\mathbb{R}\setminus[x_1,x_2]} (T(x)-x)^2 f(x)\,dx \leq \int_{\mathbb{R}\setminus[x_1,x_2]} 2(T(x)^2+x^2) f(x)\,dx \leq 2^{17}\varepsilon(\ln\varepsilon)^2.$$

Combining this estimate with (13), completes the proof of Lemma 4.1. □

5 The L_1 and quadratic transportation distances

Our goal is to estimate the L_1 distance of two log-concave probability distributions f and g in terms of their quadratic transportation distance. In this section, T always denotes the transportation map $T: I_f \to I_g$ satisfying

$$\int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{T(x)} g(t)\,dt.$$

We prepare our estimate Theorem 5.3 by Propositions 5.1 and 5.2.

When we write $A \ll B$ for expressions A and B, then we mean that $|A| \leq c \cdot B$ where $c > 0$ is an absolute constant, and hence is independent from all the quantities occurring in A and B. In addition $A \approx B$ means that $A \ll B$ and $B \ll A$.

PROPOSITION 5.1 *Let f and g be log-concave probability distributions on $\mathbb{R}$ satisfying $\int_{-\infty}^{z} f \geq \nu$ and $\int_{z}^{\infty} f \geq \nu$ for $\nu > 0$ and $z \in \mathbb{R}$. If either $\int_{-\infty}^{z} g \leq \nu/2$ or $\int_{z}^{\infty} g \leq \nu/2$, then*

$$\int_{z-\frac{\nu}{f(z)}}^{z+\frac{\nu}{f(z)}} (T(x)-x)^2 f(x)\,dx \gg \frac{\nu^3}{f(z)^2}.$$

Proof: We may assume that $\int_{z}^{\infty} g \leq \nu/2$. It follows from Corollary 2.3 (i) that if $x \leq z + \frac{\nu \ln \frac{3}{2}}{f(z)}$ then $\int_{z}^{x} f \leq \nu/2$, and hence $T(x) \leq z$. Therefore

$$\int_{z+\frac{\nu \ln \frac{5}{4}}{f(z)}}^{z+\frac{\nu \ln \frac{3}{2}}{f(z)}} (T(x)-x)^2 f(x)\,dx \gg \int_{z+\frac{\nu \ln \frac{5}{4}}{f(z)}}^{z+\frac{\nu \ln \frac{3}{2}}{f(z)}} \left(\frac{\nu \ln \frac{5}{4}}{f(z)}\right)^2 \frac{f(z)}{2}\,dx \gg \frac{\nu^3}{f(z)^2}. \quad \Box$$

PROPOSITION 5.2 *Let f and g be log-concave probability distributions on $\mathbb{R}$ satisfying $\int_{-\infty}^{z} f \geq \nu$ and $\int_{z}^{\infty} f \geq \nu$, moreover $\int_{-\infty}^{z} g \geq \nu/2$ and $\int_{z}^{\infty} g \geq \nu/2$ for $\nu > 0$ and $z \in \mathbb{R}$. If $g(z) \neq f(z)$ and $\Delta = \frac{\nu \ln 2}{3f(z)} \cdot \min\{|\ln \frac{g(z)}{f(z)}|, 3\}$, then*

$$\int_{z-\Delta}^{z+\Delta} (T(x)-x)^2 f(x)\,dx \gg \frac{\nu^3}{f(z)^2} \cdot \min\left\{\left|\ln \frac{g(z)}{f(z)}\right|, 3\right\}^4.$$

Remark If in addition $e^{-3}f(z) \leq g(z) \leq e^3 f(z)$, then the arguments in Cases 2 and 3 show that the interval $[z-\Delta, z+\Delta]$ of integration can be replaced by $[z-\frac{\Delta}{150}, z+\frac{\Delta}{150}]$, and if $x \in [z-\frac{\Delta}{150}, z+\frac{\Delta}{150}]$, then

$$\tfrac{1}{3}\,|\ln \tfrac{g(z)}{f(z)}| \leq |\ln \tfrac{g(x)}{f(x)}| \leq \tfrac{5}{3}\,|\ln \tfrac{g(z)}{f(z)}|.$$

Proof: According to Corollary 2.3 (i), if $z - \Delta \leq x \leq z + \Delta$, then

$$f(z)/2 \leq f(z) \cdot e^{\frac{-f(z)|x-z|}{\nu}} \leq f(x) \leq f(z) \cdot e^{\frac{f(z)|x-z|}{\nu}} \leq 2f(z). \tag{14}$$

Similarly if $z - \frac{\nu \ln 2}{2g(z)} \leq x \leq z + \frac{\nu \ln 2}{2g(z)}$, then

$$g(z)/2 \leq g(z) \cdot e^{\frac{-2g(z)|x-z|}{\nu}} \leq g(x) \leq g(z) \cdot e^{\frac{2g(z)|x-z|}{\nu}} \leq 2g(z). \tag{15}$$

We may assume

$$T(z) \leq z.$$

For the rest of the argument, we distinguish four cases.

Case 1 $g(z) \geq e^3 f(z)$.
In this case, $\Delta = \frac{\nu \ln 2}{f(z)}$. We note that,

$$\frac{\ln 2}{2 \cdot e^3} < \frac{\ln 2}{10} < \frac{3 \ln 2}{10} < \ln \frac{5}{4}. \tag{16}$$

Since $\frac{\nu \ln 2}{2g(z)} < \frac{\Delta}{10}$, (15) yields that if $x \geq z + \frac{\Delta}{10}$, then

$$\int_z^x g > \frac{\nu}{4}. \tag{17}$$

However (14) and (16) yield that if $z < x \leq z + \frac{3\Delta}{10}$, then

$$\int_z^x f < \frac{\nu}{4}. \tag{18}$$

Since $T(z) \leq z$, (17) and (18) yield that if $z + \frac{2\Delta}{10} \leq x \leq z + \frac{3\Delta}{10}$, then $T(x) \leq z + \frac{\Delta}{10}$. In particular

$$\int_{z+\frac{2\Delta}{10}}^{z+\frac{3\Delta}{10}} (T(x) - x)^2 f(x)\, dx \geq \int_{z+\frac{2\Delta}{10}}^{z+\frac{3\Delta}{10}} \left(\frac{\Delta}{10}\right)^2 \frac{f(z)}{2}\, dx \gg \Delta^3 f(z).$$

Case 2 $f(z) < g(z) \leq e^3 f(z)$.
Let $\lambda = (\frac{f(z)}{g(z)})^{\frac{1}{3}} \geq 1/e$. Since $2g(z) \leq 2e^3 f(z) < 50 f(z)$ and $\Delta = \frac{\nu \ln 2}{3f(z)} \ln \frac{g(z)}{f(z)}$, if $z \leq x \leq z + \frac{1}{50}\Delta$, then (14) and (15) yield

$$\lambda \cdot f(z) \leq f(x) \leq \lambda^{-1} \cdot f(z) \quad \text{and} \quad \lambda \cdot g(z) \leq g(x) \leq \lambda^{-1} \cdot g(z).$$

In particular if $z \leq s, t \leq z + \frac{1}{50}\Delta$, then $\frac{f(s)}{g(t)} \leq \lambda$. We deduce that if $z < x \leq z + \frac{1}{150}\Delta$ then

$$\int_z^x f \leq \int_z^{z+\lambda(x-z)} g.$$

Thus $T(x) \leq z + \lambda(x - z)$ by $T(z) \leq z$, and hence

$$x - T(x) \geq (1 - \lambda)(x - z) = \lambda \left(\frac{1}{\lambda} - 1\right)(x - z) \geq \frac{x - z}{3e} \cdot \ln \frac{g(z)}{f(z)}.$$

It follows that

$$\int_{z+\frac{\Delta}{300}}^{z+\frac{\Delta}{150}} (T(x)-x)^2 f(x)\,dx \gg \Delta^3 f(z) \ln \frac{g(z)}{f(z)}.$$

Case 3 $e^{-3}f(z) \le g(z) < f(z)$.
Let $\lambda = (\frac{f(z)}{g(z)})^{\frac{1}{3}} \le e$. Since $\Delta = \frac{\nu \ln 2}{3f(z)} \ln \frac{f(z)}{g(z)}$, if $z - \frac{1}{2}\Delta \le x \le z$, then (14) and (15) yield

$$\lambda^{-1} \cdot f(z) \le f(x) \le \lambda \cdot f(z) \quad \text{and} \quad \lambda^{-1} \cdot g(z) \le g(x) \le \lambda \cdot g(z).$$

In particular if $z - \frac{1}{2}\Delta \le s, t \le z$, then $\frac{f(s)}{g(t)} \ge \lambda$. We deduce that if $z - \frac{1}{2e}\Delta < x \le z$ then

$$\int_x^z f \ge \int_{z-\lambda(z-x)}^z g.$$

Thus $T(x) \le z - \lambda(z-x)$ by $T(z) \le z$, and hence

$$x - T(x) \ge (\lambda - 1)(z-x) \ge \frac{z-x}{3} \cdot \ln \frac{f(z)}{g(z)}.$$

It follows that

$$\int_{z-\frac{\Delta}{150}}^{z-\frac{\Delta}{300}} (T(x)-x)^2 f(x)\,dx \gg \Delta^3 f(z) \ln \frac{f(z)}{g(z)}.$$

Case 4 $g(z) \le e^{-3}f(z)$.
Since $\Delta = \frac{\nu \ln 2}{f(z)}$, if $z - \Delta \le x \le z$, then (14) and (15) yield that $f(x) \ge f(z)/2$ and $g(x) \le 2g(z)$, respectively. In particular if $z - \Delta \le s, t \le z$, then $f(s) \ge 2g(t)$. We deduce that if $z - \frac{1}{2}\Delta < x \le z$ then

$$\int_x^z f \ge \int_{z-2(z-x)}^z g.$$

Thus $T(x) \le z - 2(z-x)$ by $T(z) \le z$, and hence $x - T(x) \ge z - x$. It follows that

$$\int_{z-\frac{\Delta}{2}}^{z-\frac{\Delta}{4}} (T(x)-x)^2 f(x)\,dx \gg \Delta^3 f(z). \quad \square$$

THEOREM 5.3 *If f and g are log-concave probability distributions on $\mathbb{R}$, and $\int_{I_f} f(x)(T(x)-x)^2dx = \varepsilon \cdot f(w_f)^{-2}$ for $\varepsilon \in (0,1)$, then*

$$\int_{\mathbb{R}} |f(x)-g(x)|\,dx \ll \sqrt[3]{\varepsilon}|\ln\varepsilon|^{\frac{2}{3}}.$$

Remark According to Example 7.2, the exponent $\frac{1}{3}$ of ε is optimal in Lemma 5.3.

Proof: It is enough to prove the statement if $\varepsilon < \varepsilon_0$, where $\varepsilon_0 \in (0,\frac{1}{2})$ is an absolute constant specified later. We may assume that $f(w_f)=1$, and hence $f(x) \leq 2$ for any $x \in \mathbb{R}$ by Proposition 2.2 (iii), and for the inverse S of T,

$$\int_{I_f} f(x)(T(x)-x)^2dx = \int_{I_g} g(y)(S(y)-y)^2dy \leq \varepsilon.$$

For $x \in \mathbb{R}$, we define

$$\begin{aligned} \nu(x) &= \min\left\{\int_{-\infty}^{x} f, \int_{x}^{\infty} f\right\}, \\ \tilde{\nu}(x) &= \min\left\{\int_{-\infty}^{x} g, \int_{x}^{\infty} g\right\}. \end{aligned}$$

First we estimate g. Since $\nu(w_f)=\frac{1}{2}$, if ε_0 is small enough then Propositions 5.1 and 5.2 yield that $\tilde{\nu}(w_f) > \frac{1}{4}$ and $g(w_f) \leq 2$, respectively. We conclude by Proposition 2.2 (ii) that $g(w_g) \leq 4$, and hence $g(x) \leq 8$ for any $x \in \mathbb{R}$ by Proposition 2.2 (iii).

It follows by $f(x) \leq 2$ and Proposition 5.1 that there exists a positive constant c_1 such that if $\nu(x) \geq c_1\sqrt[3]{\varepsilon}$ then $\tilde{\nu}(x) \geq \nu(x)/2$. Now applying Proposition 5.1 to g, and possibly increasing c_1, we have the following: If $\nu(x) \geq c_1\sqrt[3]{\varepsilon}$ then $\tilde{\nu}(x) \leq 2\nu(x)$. Finally, possibly increasing c_1 further, if $\nu(x) \geq c_1\sqrt[3]{\varepsilon}$, then $|\ln\frac{g(x)}{f(x)}| \leq \ln 2$ by Proposition 5.2. We choose ε_0 small enough to satisfy $2c_1\sqrt[3]{\varepsilon_0} < \frac{1}{2}$.

For $z \in \mathbb{R}$, we define $\Delta(z) = \frac{\nu \ln 2}{450 f(z)} \cdot |\ln\frac{g(z)}{f(z)}|$. We assume $\nu(z) \geq c_1\sqrt[3]{\varepsilon}$, and hence $\frac{1}{2} \leq \frac{g(z)}{f(z)} \leq 2$. It follows by Corollary 2.3 (i) that and $f(x) \geq f(z)/2$ $\nu(x) \leq 2\nu(z)$ if $x \in [z-\Delta(z), z+\Delta(z)]$. We deduce using Proposition 5.2 and its remark that there exists an absolute constant c_2 such that assuming $g(z) \neq f(z)$, we have

$$\int_{z-\Delta(z)}^{z+\Delta(z)} \frac{\nu(x)^2}{f(x)} \cdot \left|\ln\frac{g(x)}{f(x)}\right|^3 dx \leq c_2 \int_{z-\Delta(z)}^{z+\Delta(z)} (T(x)-x)^2 f(x)\,dx. \qquad (19)$$

We define $z_1 < z_2$ by the properties $\nu(z_1) = \nu(z_2) = 2c_1\sqrt[3]{\varepsilon}$. We observe that if $g(z) \neq f(z)$ and some $x \in [z - \Delta(z), z + \Delta(z)]$ satisfies $\nu(x) \geq 2c_1\sqrt[3]{\varepsilon}$ then $\nu(z) \geq c_1\sqrt[3]{\varepsilon}$. It is not hard to show based on (19) that

$$\int_{z_1}^{z_2} \frac{\nu(x)^2}{f(x)} \cdot \left|\ln \frac{g(x)}{f(x)}\right|^3 dx \leq c_2 \int_{\mathbb{R}} (T(x) - x)^2 f(x)\, dx.$$

Since $f(x) \leq 2$ and $\frac{|f(x)-g(x)|}{f(x)} \leq 4|\ln \frac{g(x)}{f(x)}|$ for $x \in [z_1, z_2]$, we deduce

$$\begin{aligned}\int_{z_1}^{z_2} \frac{\nu(x)^2|f(x)-g(x)|^3}{f(x)^2}\, dx &= 4\int_{z_1}^{z_2} \frac{\nu(x)^2}{f(x)} \left(\frac{|f(x)-g(x)|}{f(x)}\right)^3 dx \\ &\leq 4^4 \int_{z_1}^{z_2} \frac{\nu(x)^2}{f(x)} \left|\ln \frac{g(x)}{f(x)}\right|^3 dx \leq 4^4 c_2 \varepsilon.\end{aligned}$$

It follows by the Hölder inequality that

$$\begin{aligned}\int_{z_1}^{z_2} |f(x) - g(x)|\, dx &= \int_{z_1}^{z_2} \frac{\nu(x)^{\frac{2}{3}}|f(x)-g(x)|}{f(x)^{\frac{2}{3}}} \cdot \frac{f(x)^{\frac{2}{3}}}{\nu(x)^{\frac{2}{3}}}\, dx \\ &\leq \left[\int_{z_1}^{z_2} \frac{\nu(x)^2|f(x)-g(x)|^3}{f(x)^2}\, dx\right]^{\frac{1}{3}} \times \\ &\quad \times \left[\int_{z_1}^{z_2} \frac{\nu(x)}{f(x)}\, dx\right]^{\frac{2}{3}}.\end{aligned}$$

Here $f(x) = |\nu'(x)|$, therefore

$$\begin{aligned}\int_{z_1}^{z_2} |f(x) - g(x)|\, dx &\leq (4^4 c_2 \varepsilon)^{\frac{1}{3}} \left[\int_{z_1}^{w_f} \frac{\nu'(x)}{\nu(x)}\, dx + \int_{w_f}^{z_2} \frac{-\nu'(x)}{\nu(x)}\, dx\right]^{\frac{2}{3}} \\ &= (4^4 c_2 \varepsilon)^{\frac{1}{3}} \left[2 \cdot \ln \tfrac{1}{2} - 2 \cdot \ln(2c_1\sqrt[3]{\varepsilon})\right]^{\frac{2}{3}} \ll \sqrt[3]{\varepsilon}|\ln \varepsilon|^{\frac{2}{3}}.\end{aligned}$$

On the other hand, $\tilde{\nu}(x_i) \leq 2\nu(x_i) = 4c_1\sqrt[3]{\varepsilon}$, $i = 1, 2$, yields that

$$\int_{-\infty}^{z_1} |f(x) - g(x)|\, dx \leq 6c_1\sqrt[3]{\varepsilon} \quad \text{and} \quad \int_{z_2}^{\infty} |f(x) - g(x)|\, dx \leq 6c_1\sqrt[3]{\varepsilon},$$

and in turn we conclude Theorem 5.3. $\square$

6 The proof of Theorem 1.2

For a non-negative, bounded, and not identically zero function h on $\mathbb{R}$, its log-concave hull is

$$\tilde{h}(x) = \inf\{p(x) :\ p \text{ is a log-concave function s.t. } h(t) \le p(t) \text{ for } t \in \mathbb{R}\}.$$

This $\tilde{h}$ is log-concave and $h(t) \le \tilde{h}(t)$ for all $t \in \mathbb{R}$, therefore we may take minimum in the definition. Next we present a definition of $\tilde{h}$ in terms of $\ln h$. Let J_h be the set of all $x \in \mathbb{R}$ with $h(x) > 0$, and let

$$C_h = \{(x, y) \in \mathbb{R}^2 :\ x \in J_h \ \text{ and } \ y \le \ln h(x)\}.$$

This C_h is convex if and only if h is log-concave. In addition $J_{\tilde{h}}$ is the convex hull of J_h, and the interior of $C_{\tilde{h}}$ is the interior of the convex hull of C_h. We also observe that for any unit vector $u \in \mathbb{R}^2$, we have

$$\sup\{\langle u, v\rangle :\ v \in C_h\} = \sup\{\langle u, v\rangle :\ v \in C_{\tilde{h}}\}. \tag{20}$$

Let f, g and m be the functions in Theorem 1.2. The condition of the Prékopa-Leindler inequality is equivalent with

$$\tfrac{1}{2}(C_f + C_g) \subset C_m, \tag{21}$$

where $C_f + C_g$ is the Minkowski sum of the two sets. Choose $x_0, y_0 \in \mathbb{R}$ such that $f(x_0) > 0$ and $g(y_0) > 0$. For any $x \in \mathbb{R}$, $m(\frac{x+x_0}{2}) \ge \sqrt{f(x_0)g(x)}$ and $m(\frac{x+y_0}{2}) \ge \sqrt{f(x)g(y_0)}$, and hence

$$f(x) \le \frac{m(\frac{x+y_0}{2})^2}{g(y_0)} \quad \text{and} \quad g(x) \le \frac{m(\frac{x_0+x}{2})^2}{f(x_0)}.$$

Since m is log-concave function with finite integral, it is bounded, thus f and g are bounded, as well. Therefore we may define the log-concave hull of f and g of $\tilde{f}$ and $\tilde{g}$, respectively. It follows that $\tilde{f}(x) \ge f(x)$ and $\tilde{g}(y) \ge g(y)$. Since m is log-concave, (20) and (21) yield that $m(\frac{x+y}{2}) \ge \sqrt{\tilde{f}(x)\tilde{g}(y)}$ for $x, y \in \mathbb{R}$. We may assume that $\tilde{f}$ and $\tilde{g}$ are probability distributions with zero mean, and $\tilde{f}(w_{\tilde{f}}) = 1$. It follows that

$$\int_{\mathbb{R}} f \ge 1 - \varepsilon, \ \int_{\mathbb{R}} g \ge 1 - \varepsilon, \ \int_{\mathbb{R}} m \le 1 + \varepsilon. \tag{22}$$

Next applying (5), Lemma 4.1 and Theorem 5.3 to $\tilde{f}$ and $\tilde{g}$, we conclude

$$\int_{\mathbb{R}} |\tilde{f}(t) - \tilde{g}(t)|\, dt \ll \sqrt[3]{\varepsilon} |\ln \varepsilon|^{\frac{4}{3}}. \tag{23}$$

In addition (22) yields

$$\int_{\mathbb{R}} |\tilde{f}(t) - f(t)|\, dt \leq \varepsilon \ \text{ and } \int_{\mathbb{R}} |\tilde{g}(t) - g(t)|\, dt \leq \varepsilon. \tag{24}$$

Therefore to complete the proof of Theorem 1.2, all we have to do is to estimate $\int_{\mathbb{R}} |m(t) - \tilde{g}(t)|\, dt$. For this, let $T : I_{\tilde{f}} \to I_{\tilde{g}}$ be the transportation map satisfying

$$\int_{-\infty}^{x} \tilde{f}(t)\, dt = \int_{-\infty}^{T(x)} \tilde{g}(t)\, dt.$$

We note that $R(x) = \frac{x+T(x)}{2}$ is an increasing and bijective map from $I_{\tilde{f}}$ into $\frac{1}{2}(I_{\tilde{f}} + I_{\tilde{g}})$. We define the function $h : \mathbb{R} \to \mathbb{R}$ as follows. If $x \notin \frac{1}{2}(I_{\tilde{f}} + I_{\tilde{g}})$, then $h(x) = 0$, and if $x \in I_{\tilde{f}}$, then

$$h\left(\frac{x + T(x)}{2}\right) = \sqrt{\tilde{f}(x)\tilde{g}(T(x))}.$$

We have $h(x) \leq m(x)$, and the proof of the Prékopa-Leindler inequality using the transportation map in Section 3 shows that $\int_{\mathbb{R}} h \geq 1$. We deduce by (22) that

$$\int_{\mathbb{R}} |m(t) - h(t)|\, dt \leq \varepsilon. \tag{25}$$

To compare h to $\tilde{g}$, we note that $\int_{\mathbb{R}} h \leq 1 + \varepsilon$ and (22) imply

$$\int_{\mathbb{R}} h(t) - \tilde{g}(t)\, dt \leq 2\varepsilon. \tag{26}$$

Let $B \subset \mathbb{R}$ be the set of all $t \in \mathbb{R}$ where $\tilde{g}(t) < h(t)$, and hence $B \subset \frac{1}{2}(I_{\tilde{f}} + I_{\tilde{g}})$. In addition let $A = R^{-1}B \subset I_{\tilde{f}}$. If $t = \frac{x+T(x)}{2} \in B$ for $x \in A$ then as $\tilde{g}$ is

concave and $\tilde{f}(x)=\tilde{g}(T(x))T'(x)$, we have

$$\begin{aligned}
[h(R(x)-\tilde{g}(R(x))]\cdot R'(x) &\leq \left[\sqrt{\tilde{f}(x)\tilde{g}(T(x))}-\sqrt{\tilde{g}(x)\tilde{g}(T(x))}\right]\cdot\frac{1+T'(x)}{2}\\
&\leq (\tilde{f}(x)-\tilde{g}(x))\cdot\frac{\sqrt{\tilde{g}(T(x))}}{\sqrt{\tilde{f}(x)}}\cdot\frac{1+T'(x)}{2}\\
&= (\tilde{f}(x)-\tilde{g}(x))\cdot\left(1+\frac{(1-\sqrt{T'(x)})^2}{2\sqrt{T'(x)}}\right).
\end{aligned}$$

In particular $\tilde{g}(x)<\tilde{f}(x)$ for $x\in A$. It follows from (5) and (23) that

$$\begin{aligned}
\int_B h(t)-\tilde{g}(t)\,dt &= \int_A [h(R(x)-\tilde{g}(R(x))]\cdot R'(x)\,dx\\
&\leq \int_{I_{\tilde{f}(x)}} |\tilde{f}(x)-\tilde{g}(x)|+\tilde{f}(x)\cdot\frac{(1-\sqrt{T'(x)})^2}{2\sqrt{T'(x)}}\,dx\\
&\ll \sqrt[3]{\varepsilon}|\ln\varepsilon|^{\frac{4}{3}}.
\end{aligned}$$

It follows from (26) that $\int_{\mathbb{R}}|h(t)-\tilde{g}(t)|\,dt\ll\sqrt[3]{\varepsilon}|\ln\varepsilon|^{\frac{4}{3}}$. Therefore combining this estimate with (24) and (25) leads to $\int_{\mathbb{R}}|m(t)-g(t)|\,dt\ll\sqrt[3]{\varepsilon}|\ln\varepsilon|^{\frac{4}{3}}$. In turn we deduce $\int_{\mathbb{R}}|m(t)-f(t)|\,dt\ll\sqrt[3]{\varepsilon}|\ln\varepsilon|^{\frac{4}{3}}$ by (23) and (24). $\square$

REMARK 6.1 *Careful check of the argument shows that the estimate for* $\int_{\mathbb{R}}|m(t)-f(t)|\,dt$ *and* $\int_{\mathbb{R}}|m(t)-g(t)|\,dt$ *is of the same order as the estimate for* $\int_{\mathbb{R}}|\tilde{f}(t)-\tilde{g}(t)|\,dt$*. Therefore to improve on the estimate in Theorem 1.2, all one needs to improve is (23).*

7 Appendix - Examples

Example 7.1 *If f is an even log-concave probability distribution, $g(x)=(1+\varepsilon)\cdot f((1+\varepsilon)x)$, and $m(x)=(1+\varepsilon)\cdot f(x)$, then we have (5), and*

$$\int_{I_f} f(x)(T(x)-x)^2dx=\frac{\varepsilon^2}{(1+\varepsilon)^2}\int_{\mathbb{R}}x^2f(x)dx.$$

Example 7.2 *Let f be the constant one on $[-\frac{1}{2}, \frac{1}{2}]$, and let g a modification such that if $|x| \geq \frac{1}{2} - \varepsilon$ then*

$$g(x) = e^{-\frac{|x| - \frac{1}{2} + \varepsilon}{\varepsilon}}.$$

In addition

$$m(x) = \begin{cases} 1 & \text{if } x \in [-\frac{1}{2}, \frac{1}{2}] \\ e^{-\frac{|x| - \frac{1}{2}}{\varepsilon}} & \text{otherwise.} \end{cases}$$

In this case $\int_{\mathbb{R}} m = 1 + \varepsilon$,

$$\int_{\mathbb{R}} f(x) \cdot \frac{(1 - \sqrt{T'(x)})^2}{2\sqrt{T'(x)}} \, dx \approx \varepsilon \quad \text{and} \quad \int_{\mathbb{R}} |f(x) - g(x)| \, dx \approx \varepsilon.$$

Moreover $\int_{\mathbb{R}} f(x)(T'(x) - 1)^2 dx = \infty$ and $\int_{\mathbb{R}} f(x)(T(x) - x)^2 dx \approx \varepsilon^3$.

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